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A GENERAL HOBBY–RICE THEOREM AND CAKE CUTTING

BY

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ABSTRACT

Let X be a Borel subset of a separable Banach space E. Let μ be a non-atomic, σ-finite, Borel measure on X. Let $G \subseteq L_1(X, \Sigma, \mu)$ be m-dimensional.

THEOREM: *There is an $l \in E^*$ and real numbers $-\infty = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty$ with $n \leq m$, such that for all $g \in G$,*

$$\sum_{i=0}^{n} (-1)^i \int_{X \cap l^{-1}[x_i, x_{i+1}]} g \, d\mu = 0.$$

1. Introduction

Let G be an m-dimensional subspace of Lebesgue integrable functions on $[0, 1]$. The Hobby–Rice Theorem [4] states that there are real numbers $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$, with $n \leq m$, such that for all $g \in G$,

$$\sum_{i=0}^{n} (-1)^i \int_{[x_i, x_{i+1}]} g \, d\mu = 0,$$

where μ represents Lebesgue measure. A particularly nice proof of the theorem was found by A. Pinkus [7], and applications of the theorem in L_1 approximation are given in his book [8].

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The Hobby–Rice Theorem for Lebesgue measure on $[0,1]$ will be extended to any non-atomic, σ-finite, Borel measures on a measurable subset, X, of a separable Banach space, E. The division of $[0,1]$ by the points $\{x_i\}$ in the Hobby–Rice Theorem is replaced by parallel slicings of X by hyperplanes of the form $l^{-1}(x_i)$ for some $l \in E^*$.

This paper also contains two other classes of results. One is used in the proof of the generalization, the other illustrates applying the generalization to $E = \mathbf{R}^n$.

The first type of results involves constructing an $l \in E^*$ having all level sets, $l^{-1}(a)$, of measure zero. Proving the existence of such an l is a significant step in our argument. We prove its existence in an abstract setting for a σ-finite, non-atomic measure space (X, Σ, μ). That is, X need not be a subset of a linear space. The general result shows that there are functions, f, such that for all real a, $\mu(f^{-1}(a)) = 0$, for f's belonging to each of the following classes:

(i) $L_p(X, \Sigma, \mu)$,

(ii) $C(X)$,

(iii) the lower semicontinuous functions on X,

(iv) E^*, for $X \subseteq E$, and

(v) the positive linear functionals on E.

The classes require appropriate settings. For example, the continuous and semicontinuous functions require a topology on X, and the positive functionals require a lattice structure on E. For the continuous functions, we also need the measure to be a non-atomic Baire measure.

The existence of such functions provides a strengthening of the one measure version of the Liapanov Theorem. For example it shows:

PROPOSITION: *If (X, Σ, μ) is a finite, non-atomic, Borel measure space, then there is a nested collection of open sets $\{U_i\}_{i \in I}$ with $I = [0, \mu(X)]$ such that*

(i) *$i < j$ implies that $U_i \subseteq U_j$, and*

(ii) *for $i \in I$, $\mu(U_i) = i$.*

Furthermore, if μ is also non-atomic as a Baire measure, the sets $\{U_i\}_{i \in I}$ can have the additional properties:

(iii) *$U_i \in \{\operatorname{supp} f : f \in C^+(X)\}$,*

(iv) *$i < j$ implies that $\operatorname{cl} U_i \subseteq U_j$, and*

(v) *$\mu(U_i) = \mu(\operatorname{cl} U_i)$.*

In addition, if X is a measurable subset of a separable Banach space E,

(vi) *the sets U_i can be taken to be linear half spaces.*

The second type of results comprises applications to a setting called *cake slicing problems*. The problems are variations of the following 3-dimensional model.

A cake is to be divided with parallel, straight slices and distributed to a finite number of parties. The equity of the distribution is assessed by a panel of m judges. Each judge has his/her own measure of the value for all portions of the cake. We require that each judge be satisfied with the fairness of the partitioning of the cake.

General presentations of cake cutting and related problems are in Dubins and Spannier [1] and in Karlin and Studden [5, chapter VIII sect 14].

The outline of the paper is as follows. Section 3 contains preliminary results and continues to introduce definitions, and hypotheses used in the paper. Sections 4 and 5 culminate in a theorem proving, in a general setting, the existence of functions having all level sets of measure zero. Section 6 contains the main results. Its first theorem applies the general result of Section 5 to six specific settings using three different norms. It then applies these results on zero measure level sets, to prove a strengthened Liapanov's Theorem. The linear half space portion of this theorem is then used to prove the generalized Hobby–Rice Theorem. Section 7 gives three cake cutting applications.

The paper assumes no regularity conditions on the measure space or on the topology.

2. Notation and definitions

MEASURE SPACES. Throughout the paper (X, Σ, μ) will be a measure space. We use *a.e.* as an abbreviation for the term "almost everywhere". We will sometimes write L_1 for $L_1(X, \Sigma, \mu)$. If S is a collection of subsets of X, the σ-algebra generated by S is the smallest σ-algebra of sets that contains S. If there is also a topology on X, a set in the smallest σ-algebra containing the open sets is a **Borel** set. The σ-algebra generated by the supports of the nonnegative, continuous real function is the **Baire** sets.

For $U, K \in \Sigma$, we say that U **splits** K if $\mu(K) > \mu(K \cap U) > 0$. A set $K \in \Sigma$ is an **atom** of μ if $\mu(K) > 0$, and no U in Σ splits K. (X, Σ, μ) is **non-atomic** if no set in Σ is an atom with respect to μ.

FUNCTIONS. Let f be a real-valued function defined on X. The **support** of f is $\text{supp}(f) := \{x: f(x) \neq 0\}$. The **zero** set of f is $Zf = Z(f) := \{x: f(x) = 0\}$. The sign of a function f is

$$\operatorname{sgn} f(x) := \begin{cases} 1 & \text{if } f(x) > 0; \\ -1 & \text{if } f(x) < 0; \\ 0 & \text{if } f(x) = 0. \end{cases}$$

We will use f^+ and f^- to be $\max\{f(x), 0\}$ and $-\min\{f(x), 0\}$ respectively. The **characteristic** or **indicator** function of a set $A \subseteq X$ is

$$I_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

SPACES OF FUNCTIONS. The continuous functions are written $C(X)$, and $C^+(X) = \{f \in C(X) : 0 \leq f\}$. The linear span of a collection of functions, F, is $\operatorname{span} F$. The dual of a Banach space E is written E^*.

SETS. For sets K and U, we use $K - U$ to represent $\{x \in K\colon x \notin U\}$, and the **complement** of K is $K^c := X - K$. The closure of a set K is written $\operatorname{cl} K$.

3. Preliminaries

We will later use the following fact about non-atomic measure spaces.

LEMMA 3-1: *Let (X, Σ, μ) be non-atomic. Let $S \subseteq \Sigma$ generate Σ. Let K be a set in Σ such that $\infty > \mu(K) > 0$. There is a $U \in S$ that splits K.*

Proof: $M = \{V \in \Sigma\colon V$ does not split $K\}$ is a σ-algebra. If $S \subseteq M$, then K would be an atom. ∎

RESERVED NOTATION. In this paper we will reserve L for a convex cone of functions defined on X. That is, if f and g are in L, then $sf + rg \in L$ for s and r nonnegative real numbers. We define the **half spaces associated with** L to be $\{f^{-1}(0, \infty)\colon f \in L\}$. If L is a subset of continuous linear functionals, these sets are called **linear half spaces**. Notice that if f is a linear functional, what we call a linear half space corresponds with the usual terminology, but if f is an arbitrary continuous function our "half space" corresponds with the support of f^+.

In the next sections we will prove the existence of functions having level sets of measure zero. We will prove this for convex cones of functions from three different normed spaces — the uniform norm, the L_p-norm, and the dual space norm.

We isolate two common properties of these norms, and assume that we have any norm satisfying the two properties. Being so general, these properties appear to be technical. However, we only directly use this abstract norm to prove one result — Lemma 4-3.

Our first step — Lemma 3-2 — will be to show that the three norms referred to possess the two needed properties.

SETTING. L is contained in (or equal to) a normed linear space of real functions defined on X.

HYPOTHESIS. We require that the norm, when restricted to L, has two properties.

Let f_i converge to f in norm. For $\{a_i\} \subseteq \mathbf{R}$, and $a_i \to a$, we require:

(i) for $\eta > 0$ and $\epsilon > 0$ there is an M such that if $i \geq M$,

$$\mu\big(f_i^{-1}(a_i) - f^{-1}(a-\eta, a+\eta)\big) < \epsilon;$$

(ii) for $\epsilon > 0$ there are integers k and M such that for $i \geq M$,

$$\mu(f_i^{-1}(k, \infty)) < \epsilon.$$

LEMMA 3-2: *The following norms and sets L satisfy the properties hypothesized above.*

(a) *L is a bounded family of functions defined on X equipped with the uniform norm.*

(b) *X is a Borel subset of a Banach space E, $\mu(X) < \infty$, and $L \subseteq E^*$ has the dual space norm.*

(c) *$L \subseteq L_p(X, \Sigma, \mu)$, for $1 \leq p \leq \infty$.*

Proof: We prove the three cases individually.

Proof of (a), the uniform norm: Let M be large enough that $i \geq M$ implies both $\|f_i - f\| < \frac{\eta}{2}$, and $|a_i - a| < \eta/2$. Then for $x \in f_i^{-1}(a_i)$,

$$|f(x) - a| \leq |f(x) - f_i(x)| + |f(x) - a| \leq \|f - f_i\| + |a_i - a| < \eta,$$

and so

$$f_i^{-1}(a_i) \subseteq f^{-1}(a-\eta, a+\eta).$$

The second property follows since the sequence is bounded. That is, if $\sup\{\|f_i\|\} < B$, then for all i, $f_i^{-1}(B, \infty) = 0$.

Proof of (b), the dual space norm on E^*: Since $\mu(X) < \infty$,

$$\mu(X \cap \{x \in E : \|x\| \leq n\}) \to \mu(X) \quad \text{as } n \to \infty.$$

Choose n large enough that

$$\mu(X \cap \{x \in E : \|x\| \leq n\}) \geq \mu(X) - \epsilon/2,$$

and put

$$K = X \cap \{x \in E : \|x\| \leq n\}.$$

Since $f_i \to f$ uniformly on K, we have from the first part of this proof that eventually

$$f_i^{-1}(a_i) \cap K \subseteq f^{-1}(a-\eta, a+\eta) \cap K,$$

and property (i) follows.

To prove property (ii) we again reduce the setting to part (a) of this lemma. Let K be the set of the last paragraph. Since $f_i \to f$ uniformly on K, we have that for large i and sufficiently large k,

$$f_i^{-1}(k,\infty) \cap K = \emptyset, \quad \text{and so } \mu(f_i^{-1}(k,\infty)) < \epsilon.$$

Proof of (c), the L_p norms: We first observe that the L_∞ case follows from part (a) proved above. Otherwise suppose that $|a_i - a| < \eta/2$. Then

$$\begin{aligned} \|f_i - f\|_p^p = \int_X |f_i - f|^p d\mu &\geq \int_{f_i^{-1}(a_i) - f^{-1}(a-\eta,a+\eta)} |f_i - f|^p d\mu \\ &\geq (\eta/2)^p \mu[f_i^{-1}(a_i) - f^{-1}(a-\eta, a+\eta)]. \end{aligned}$$

So $f_i \to f$ in norm implies that condition (i) is satisfied.

To verify condition (ii), choose k so that $\mu(f^{-1}(k,\infty)) < \epsilon/2$. Then

$$\begin{aligned} \|f_i - f\|_p^p &\geq \int_X |f_i - f|^p d\mu \\ &\geq \int_{f_i^{-1}(k+1,\infty) - f^{-1}(k,\infty)} |f_i - f|^p d\mu \\ &\geq \int_{f_i^{-1}(k+1,\infty) - f^{-1}(k,\infty)} |k+1-k|^p d\mu \\ &= \mu(f_i^{-1}(k+1,\infty) - f^{-1}(k,\infty)) \\ &\geq \mu(f_i^{-1}(k+1,\infty)) - \epsilon/2. \end{aligned}$$

Since $\|f_i - f\|_p^p \to 0$, we have that for large enough i, $\mu(f_i^{-1}(k+1,\infty)) < \epsilon$. ∎

4. Level sets

LEMMA 4-1: *Let X be σ-finite. Let f and g be measurable functions. Except for possibly countably many values of r, every level set of $f + rg$ is contained, almost everywhere, in the intersection of a level set of f and a level set of g.*

Proof: This result for $\mu(X) < \infty$ is proven in the previous paper Wulbert [12].

If X is σ-finite, but not finite, then there are disjoint sets, $\{U_i\}_{i=1}^{\infty}$, of finite, positive measure that union to X. Then

$$\nu(A) = \bigcup_{i=1}^{\infty} \frac{\mu(A \cap U_i)}{2^i \mu(U_i)}$$

is a finite nonatomic measure on Σ. Since μ and ν have the same null sets, the conclusion for the level sets in (X, Σ, μ) follows from the theorem when the level sets are in (X, Σ, ν).

We alert the reader that there is a misplaced quantifier in the proof of the above-referenced Lemma 4.1 in Wulbert [12]. The sentence there should read, "Since $\mu(X) < 0$, there are at most countably many pairs of real numbers (r, a) for which...". ∎

COROLLARY 4-2: *Let X be σ-finite. Let W be a convex set in some normed linear space of functions defined on X. Let $V = \{f \in W\colon \mu(f^{-1}(a)) = 0$ for all $a \in \mathbf{R}\}$. If $V \neq \emptyset$, then V is dense in W.*

Proof: Suppose $f \in V$ and $g \in W$. From Lemma 4-1 there are $r_n \to \infty$ such that the level sets of $f + r_n g$ are contained a.e. in those of f, and hence

$$\frac{1}{1+r_n} f + \frac{r_n}{1+r_n} g$$

are functions in V that converge to g. ∎

Our only direct use of the special norm properties hypothesized above is in the proof of the following lemma.

LEMMA 4-3: *Let $\mu(X) < \infty$. For $c \geq 0$,*

$$\{f \in L\colon \text{ there is an } a \in \mathbf{R} \text{ such that } \mu(f^{-1}(a)) \geq c\}$$

is norm closed.

Proof: Let $\{f_i\} \subseteq L$ converge in norm to $f \in L$. Suppose that there are a_i such that $\mu(f_i^{-1}(a_i)) \geq c$. From property (ii) required of the norm, we may assume that the numbers a_i are bounded, and converge to a number, say a. Let $\epsilon > 0$, and let k be an integer. From norm property (i), we have that for large i,

$$\mu(f^{-1}(a - 1/k, a + 1/k)) \geq \mu(f_i^{-1}(a_i)) - \epsilon \geq c - \epsilon.$$

Hence,

$$\mu(f^{-1}(a)) = \lim_{k \to \infty} \mu(f^{-1}(a - 1/k, a + 1/k)) \geq c - \epsilon.$$

Since this is true for all $\epsilon > 0$, the proof is completed. ∎

5. Common level set

GENERAL HYPOTHESIS. Let (X, Σ, μ) be a measure space. For the remainder of the paper, we will further restrict the notation, L, to represent a convex cone of functions defined on X that is complete in a norm satisfying the norm conditions previously hypothesized.

PROPOSITION 5-1: *Let $\mu(X) < \infty$. If every $f \in L$ has a level set of measure exceeding $c_o \geq 0$ then there is a $K \in \Sigma$ with $\mu(K) > c_o$, and such that every $f \in L$ is constant, a.e., on K.*

Proof: The Baire Category Theorem and Lemma 4-3 provide that there is an integer N such that if $c = c_o + 1/N$, then

$$A = \{f \in L\colon f \text{ has a level set of measure } \geq c\}$$

has interior in L. In particular, there is an open sphere of L that is contained in A. Furthermore, the convex cone, V, generated by this sphere (i.e., all positive multiples) is also contained in A.

Let $f \in V$. There is a finite set of real numbers $\{a_1, a_2, \ldots, a_k\}$ such that $\mu(f^{-1}(a_i)) \geq c$.

Suppose there is an index "j" and a $g \in V$ such that every level set of g intersects $f^{-1}(a_j)$ in a set of measure less than c. From Lemma 4-1, there is an r such that each level set of $f + rg$ is contained, a.e., in the intersection of a level set of f and a level set of g. Hence $f + rg$ has at most $k - 1$ level sets of measure at least c. Now we replace f with $f + rg$ and repeat this process. In less than k repetitions we obtain an $f_0 \in V$, having level sets $\{f_0^{-1}(a_i)\}_{i=1}^{k}$ (we use the same letter "k") such that if $g \in V$, then for each $i = 1, 2, \ldots, k$ there are real numbers d_i, such that $\mu(f_0^{-1}(a_i) \cap g^{-1}(d_i)) \geq c$. By choosing such an f_0 for which k is maximal we also have that each d_i is unique (otherwise we could reapply Lemma 4-1 to contradict the maximality of k).

Now let

$$c^* = \inf\{\mu[f_0^{-1}(a_i) \cap g^{-1}(d)] : g \in V, d \o \mathbf{R}, \text{ and } \mu[f_0^{-1}(a_i) \cap g^{-1}(d)] \geq c\}.$$

We have that $c^* \geq c$. Choose $\{g_j\} \subseteq V$ and a unique corresponding $\{d_j\} \subseteq \mathbf{R}$ so that

$$\mu[f_0^{-1}(a_1) \cap g_j^{-1}(d_j)] \downarrow c^*.$$

Let

$$K = \bigcap_{j=1}^{\infty} [f_0^{-1}(a_1) \cap g_j^{-1}(d_j)].$$

We want to show that K is a set satisfying the statement of the proposition. We first show that $\mu(K) = c^*$.

From Lemma 4-1

$$f_0^{-1}(a_1) \cap [\bigcap_{j=1}^{w} g_j^{-1}(d_j)]$$

is a level set of $f_0 + r_1 g_1 + r_2 g_2 + \cdots + r_w g_w \in V$ for some set of positive real numbers $\{r_1, r_2, \ldots, r_w\}$. Hence,

$$c^* \leq \mu\left\{ f_0^{-1}(a_1) \cap \left[\bigcap_{j=1}^{w} g_j^{-1}(d_j) \right] \right\} \leq \mu\{f_0^{-1}(a_1) \cap g_w^{-1}(d_w)\} \to c^*.$$

Since $K = f_0^{-1}(a_1) \cap [\bigcap_{j=1}^{\infty} g_j^{-1}(d_j)]$, we have that $\mu(K) = c^*$.

Now we show that every $h \in V$ is constant, a.e. on K. There is a unique real number b for which $\mu[f_0^{-1}(a_1) \cap h^{-1}(b)] \geq c^*$. We wish to show that $\mu[h^{-1}(b) \cap K] \geq c^*$. Again applying Lemma 4-1, there are positive real numbers $\{t_0, t_1, t_2, \ldots, t_v\}$ such that

$$f_0^{-1}(a_1) \cap h^{-1}(b) \cap \left[\bigcap_{j=1}^{v} g_j^{-1}(d_j) \right]$$

is a level set of $f_0 + t_0 h + t_1 g_1 + t_2 g_2 + \cdots + t_v g_v \in V$. It, in fact, is the unique level set that intersects $f_0^{-1}(a_1)$ in a set of measure at least c^*. So

$$c^* \leq \mu\left\{ f_0^{-1}(a_1) \cap h^{-1}(b) \cap \left[\bigcap_{j=1}^{\infty} g_j^{-1}(d_j) \right] \right\} = \mu(h^{-1}(b) \cap K) \leq c^*.$$

Hence h assumes the value b, a.e., on K.

We have shown that every function in V is constant, a.e., on K. It follows that every function in the linear span of V is constant, a.e., on K. Since V contains an open sphere of L, this span of V contains the convex cone L. That is, suppose that $v \in V$ is the center of this open sphere, and $g \in L$. Then for sufficiently small $\lambda > 0$, $\lambda g + (1 - \lambda)v \in V$, and

$$g = \frac{1}{\lambda}[\lambda g + (1 - \lambda)v] + \left(1 - \frac{1}{\lambda}\right)v \in \operatorname{span} V. \quad \blacksquare$$

THEOREM 5-2: *Let $S = \{f^{-1}(0,\infty), f \in L\}$ generate Σ. Let (X,Σ,μ) be a σ-finite, nonatomic measure space. There is an $f \in L$ such that $\mu(f^{-1}(a)) = 0$ for all $a \in \mathbf{R}$.*

Proof: First suppose that $\mu(X) < \infty$. If no such f exists, then from Proposition 5-1, there is a $K \in \Sigma$ such that $\mu(K) > 0$, and such that every function in L is constant, a.e., on K. From Lemma 3-1, the half-space of some function in L splits K, and this is a contradiction.

If X is σ-finite, but not finite, then there are disjoint sets, $\{U_i\}_{i=1}^{\infty}$, of finite measure that union to X. Then

$$\nu(A) = \bigcup_{i=1}^{\infty} \frac{\mu(A \cap U_i)}{2^i \mu(U_i)}$$

is a finite nonatomic measure on Σ. If K_i are sets in Σ, then $\mu(K_i) \to 0$, if and only if $\nu(K_i) \to 0$. Hence if a norm on L satisfies the two norm conditions (hypothesized in Section 4) with respect to μ, then it also satisfies those conditions relative to ν. Therefore, from the paragraph above, there is an $f \in L$ such that $\nu(f^{-1}(a)) = 0$ for all $a \in \mathbf{R}$. But μ and ν have the same null sets. ∎

6. Specific settings

The theorem below lists common spaces that satisfy the hypothesis of Theorem 5-2.

THEOREM 6-1: *Let (X,Σ,μ) be a σ-finite, non-atomic, measure space. In each of the following settings there is an $f \in L$ such that $\mu(f^{-1}(a)) = 0$ for all $a \in \mathbf{R}$.*

(i) *L is the bounded measurable functions,*

(ii) *$L = L_p(X,\Sigma,\mu)$ for $1 \le p \le \infty$,*

(iii) *$L = C(X)$, and Σ is the Baire sets.*

For the remaining three settings let (X,Σ,μ) also be a Borel measure space.

(iv) *L is the cone of lower-semicontinuous functions,*

(v) *X is a Borel subset of a separable Banach space, E, and $L = E^*$,*

(vi) *X is a compact subset of a Banach space, E, and $L = E^*$.*

COMMENTS. Although (i) is well known from traditional measure theory, (ii) appears to be new. The case for continuous functions in (iii) was proved with a different argument in Wulbert [12]. Hans Kellerer [6] has proven (v) when $E = \mathbf{R}^n$. To show that the linear half spaces S generate Σ in (v), we use the

Hahn–Banach separation theorems to show that all open spheres are in the σ-algebra generated by the half spaces. Since the closed linear span of a compact set is a separable space, (vi) follows from (v). ∎

The corollary below is a strengthening of the one measure Liapanov Theorem, but we first need the following lemma.

LEMMA 6-2: *Let (X, Σ, μ) be a measure space. Suppose $f \in L$ is such that $\mu(f^{-1}(a)) = 0$ for all $a \in \mathbf{R}$. If $K \in \Sigma$ and $\infty > \mu(K) \geq \alpha \geq 0$, then there is an $a \in \mathbf{R}$ such that $\mu(K \cap f^{-1}(a, \infty)) = \alpha$.*

Proof: Let

$$a = \sup\{b \in \mathbf{R} : \mu[K \cap f^{-1}(b, \infty)] \geq \alpha.$$

We have that

$$\begin{aligned} \alpha \geq \lim_{n \to \infty} \mu(K \cap f^{-1}(a - 1/n, \infty)) &= \mu(K \cap f^{-1}[a, \infty)) \\ &= \mu(K \cap f^{-1}(a, \infty)) \geq \alpha. \end{aligned}$$ ∎

Applying Lemma 6-2, and the corresponding parts of Theorem 6-1 we obtain:

COROLLARY 6-3: *Let (X, Σ, μ) be a σ-finite, non-atomic, Borel measure space. There is a nested collection of open sets $\{U_i\}_{i \in I}$ such that*

(i) *$i < j$ implies that $U_i \subseteq U_j$,*

(ii) *if $K \in \Sigma$ and $\infty > \mu(K) \geq \alpha \geq 0$, then there is a unique $i \in I$ such that $\mu(K \cap U_i) = \alpha$.*

Furthermore, if μ be also non-atomic as a Baire measure, the sets $\{U_i\}_{i \in I}$ have the additional properties:

(iii) *$U_i \in \{\operatorname{supp} f : f \in C^+(X)\}$,*

(iv) *$i < j$ implies that $\operatorname{cl} U_i \subseteq U_j$, and*

(v) *$\mu(U_i) = \mu(\operatorname{cl} U_i)$.*

In addition, if X is a Borel subset of a separable Banach space E,

(vi) *the sets U_i can be taken to be linear half spaces associated with an $l \in E^*$.*

We now prove the generalized Hobby–Rice Theorem. We use the following lemma from Wulbert [12].

LEMMA 6-4: *Let (X, Σ, μ) be a measure space. Let G be an m-dimensional subspace of $L_1(X, \Sigma, \mu)$. Let Q be an $(m+1)$-dimensional, linear space of bounded measurable functions such that $\mu(Zq) = 0$ for all $q \in Q$. There is a $q \in Q$ such that for all $g \in G$,*

$$\int_X g(\operatorname{sgn} q) d\mu = 0.$$

THEOREM 6-5: *Let X be a Borel subset of a separable Banach space, E. Let μ be a σ-finite, non-atomic, Borel measure on X. Let G be an m-dimensional subspace of $L_1(X, \Sigma, \mu)$. Let $l \in E^*$ be a functional (from Theorem 6-1 part (v)), with level sets of measure zero.*

There are real numbers $-\infty = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty$ with $n \le m$, such that for all $g \in G$,

$$\sum_{i=0}^{n}(-1)^i \int_{X \cap l^{-1}(x_i, x_{i+1})} g d\mu = 0.$$

Proof: Let

$$Q = \{p(\arctan l(x)) \colon p \text{ a polynomial of degree } \le m\}.$$

If $q \in Q - \{0\}$, then Zq consists of the union of at most m level sets of l, and hence $\mu(Zq) = 0$. From Lemma 6-4, there is a $q \in Q - \{0\}$ such that $\int_X g(\operatorname{sgn} q) d\mu = 0$. Let $\{y_i\}_{i=1}^n$ be the zeros of q at which q changes sign. Let $x_i = \arctan y_i$. Then $\sum_{i=0}^n (-1)^i \int_{l^{-1}(x_i, x_{i+1})} g d\mu = 0$. ∎

Example: There is a non-atomic Borel measure (not σ-finite) on the compact set $B = \{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 \le 1\}$ such that every $l \in (\mathbf{R}^2)^*$ has a level set of measure 1.

Proof: For each $b = (x, y)$, such that $x^2 + y^2 = 1$, let μ_b be one-dimensional Lebesgue measure on the line segment, I_b, connecting b and the origin. For $K \in \Sigma$, put

$$\mu(K) = \sum_{|b|=1} \mu_b(K \cap I_b).$$

Then μ is a non-atomic measure on Σ. Furthermore, if $l \in (\mathbf{R}^2)^*$, then Zl contains I_b for some b. ∎

COMMENT. There also are non-σ-finite Borel measures on $\mathbf{R}^2$ that admit continuous linear functionals with level sets of measure zero. For example, we can construct a measure similar to the above. Except, B would be the unit square, and the μ_b's would be Lebesgue measure on the vertical line segments. Then, excluding multiples of the functional $f(x, y) = x$, all linear functionals have level sets of measure zero.

7. How to slice a cake

A region X, called a cake, is to be divided into a finite number of parts. The equity of the partitioning is to be assessed by a panel of judges. Each judge assigns his/her own measure of value, $\mu_i(K)$, to each subregion, K, of the cake. Each judge is to be satisfied.

The simplest case, perhaps, is that the cake is to be cut and distributed to m people so that each receives $1/m$ of the worth of the cake in his/her own estimation (i.e., $\frac{1}{m}\mu_i(X)$). The solution idea is to move a knife across the cake until one person declares that their fair share has been delineated. That portion is separated and given to the satisfied person. All others feel that, by their own estimates, at least $(m-1)/m$ of the value of the cake remains. The knife moving process continues cutting off additional pieces of cake until the last person is satisfied.

However, there could be complications. For example, it is possible some judges only value particular lines or planes of filling. For some orientations of the knife, such subregions would be either entirely on one side of the cut or obliterated by the cut. Corollary 6-3(vi) shows that, in fact, there are initial positionings of the knife such that the subsequent parallel moving of the blade across the cake while making slices at the satisfaction points of the judges (as described in the paragraph above) will divide the cake fairly for each participant. The cake being divided need not be any special shape (e.g., convex or connected) as long as it is measurable.

Results special to this paper are: (i) that the cuts are all "straight-line" slices. That is, the cake is cut by separating the pieces on each side of a plane (or for higher dimensions, on each side of a hyperplane — a level set of a continuous linear functional l); and (ii) all slices of the cake are parallel to each other (that is, the hyperplane "slices" are different level sets of the same continuous linear functional l). We will use the word "slice" for these divisions of the cake by planes (and hyperplanes).

To apply the results of the paper, we use:

RESERVED NOTATION. For the remainder of the paper let X be a Borel subset of a separable Banach space E. Let $\mu_1, \mu_2, \dots, \mu_m$ be non-atomic Borel measures on X of total variation 1, and put $\mu = \mu_1 + \mu_2 + \cdots + \mu_m$. Finally, $l \in E^*$ will be a functional whose level sets have μ-measure zero.

Since each μ_i is a finite, non-atomic, Baire measure, μ inherits these properties. Each μ_i is absolutely continuous with respect to μ, and therefore can be represented as $\mu_i(K) = \int_X I_K g_i d\mu$ for some $g_i \in L_1(X, \Sigma, \mu)$.

For the cake division paradigm above, we apply Theorem 6-1(v) to the measure μ to produce a continuous linear functional l with level sets of measure zero. It also has level sets of μ_i measure zero. From Lemma 6-2 there are real numbers a_i such that for each i, $\mu_i(X \cap l^{-1}(a_i, \infty)) = 1/m$. Choose the largest of these a_i, say a_k. Make the "slice" at $l^{-1}(a_k)$. Distribute the portion $X \cap l^{-1}(a_k, \infty)$ to the participant "k". If more than one participant has $\mu_i(X \cap l^{-1}(a_k, \infty)) = 1/m$, then distribute this first portion to any one of them at random. Cut a second parallel slice by applying the above process to $X - l^{-1}(a_k, \infty)$ instead of to X. Continuing the induction we have proven

THEOREM 7-1: *There are numbers, $-\infty = a_1 < a_2 < \cdots < a_m < a_{m+1} = \infty$, such that there is a one-to-one pairing between the m measures, $\{\mu_i\}$, and the m sets, $U_j = X \cap l^{-1}(a_j, a_{j+1})$, for which $\mu_i(U_j) \geq 1/m$.*

For the second application (called the bisection problem), a cake is to be distributed to two parties. The division must satisfy each of m judges. Theorem 6-5 provides that the cake can be divided with m (or fewer) parallel slices, so that giving alternate pieces to the two parties will result in all m judges assessing that the distribution is exactly equitable. That is, there exist real numbers, $-\infty = x_0 < x_1 < \cdots < x_n < x_{n+1} = \infty$, with $n \leq m$, such that for $j = 1, 2, \ldots, n$,

$$\begin{aligned}\mu_j\left(\bigcup_{i=0}^{[n/2]} l^{-1}(x_{2i}, x_{2i+1})\right) &= \sum_{i=0}^{[n/2]} \int_X I_{l^{-1}(x_{2i}, x_{2i+1})} g_j \, d\mu \\ &= \sum_{i=0}^{[n/2]} \int_X I_{l^{-1}(x_{2i+1}, x_{2i+2})} g_j \, d\mu \\ &= \mu_j\left(\bigcup_{i=0}^{[n/2]} l^{-1}(x_{2i+1}, x_{2i+2})\right),\end{aligned}$$

where $[n/2]$ is to be interpreted as the greatest integer less than or equal to $n/2$. A formal statement of this would be:

THEOREM 7-2: *There are numbers, $-\infty = x_0 < x_1 < \cdots < x_n < x_{n+1} = \infty$, with $n \leq m$, such that if*

$$U = \bigcup_{i=0}^{[n/2]} l^{-1}(x_{2i}, x_{2i+1}) \quad \text{and} \quad V = \bigcup_{i=0}^{[n/2]} l^{-1}(x_{2i+1}, x_{2i+2}),$$

then $\mu_i(U) = \mu_i(V)$ for $i = 1, 2, \ldots, m$.

Example: If the cake contains the line segment $[0, m]$, and each μ_k is Lebesgue measure on $[k-1, k]$, an equitable bisection requires m slices.

Definition: $\{U_i\}_{i=1}^p$ is a partition of X by l if: (i) $\{U_i\}_{i=1}^p$ is a partition of X, and (ii) each U_i is a union of a finite number of sets of the form $l^{-1}(x, y)$.

Our last result will show that either there is a partition of X by l into m pieces, $\{U_i\}_{i=1}^m$, such that every participant believes is exactly equitable (i.e., $\mu_i(U_j) = 1/m$ for all i and for all j); or there is a partition by l that each participant believes gives him/her more than a fair share (i.e., $\mu_i(U_i) > 1/m$ for all i).

The proof for the first lemma below is an adaptation from Karlin and Studden [5].

Lemma 7-3: *If $r_i \geq 0$ are, m, rational numbers such that $\sum_{i=1}^m r_i \leq 1$, then there is a partition, $\{U_i\}_{i=1}^m$, of X by l such that $\mu_i(U_i) \geq r_i$.*

Proof: Let M be the common denominator of the r_i. Then $r_i = n_i/M$ for integers n_i. For each i, define n_i measures identical to μ_i. This gives a total of $\sum_{i=1}^m n_i$ measures, $\{\nu_j\}$. We apply Theorem 7-1 to find points a_j so that $\nu_j(X \cap l^{-1}(a_j, a_{j+1})) \geq 1/M$. We take the union U_i of the sets associated with the n_i measures ν_j identical to μ_i. Then

$$\mu(U_i) \geq n_i 1/M \geq r_i. \qquad \blacksquare$$

Lemma 7-4: *If there is an a such that $\mu_1(X \cap l^{-1}(a, \infty)) \neq \mu_2(X \cap l^{-1}(a, \infty))$, then there is a partition, $\{U_i\}_{i=1}^m$, of X by l such that $\mu_i(U_i) > 1/m$.*

Proof: First we show that there is a set $P = l^{-1}(b, c)$ with $b < c$ such that $\mu_1(P) \neq \mu_2(P)$ and $\max \mu_i(P) < 1/m$. For any n there exists a j such that

$$\mu_1\Big(l^{-1}\Big(a + \frac{j}{2^n}, a + \frac{j+1}{2^n}\Big)\Big) \neq \mu_2\Big(l^{-1}\Big(a + \frac{j}{2^n}, a + \frac{j+1}{2^n}\Big)\Big).$$

Repeating this observation to further subdivide such sets on which $\mu_1 \neq \mu_2$, we construct a nested sequence of such sets. Their intersection is a level set of l (which has μ-measure zero). Hence there are sets in the sequence with the properties required of P.

Now let

$$Mx = \max\{\mu_i(P) : i = 1, 2, \ldots, m\}, \quad K = \{i \colon \mu_i(P) = Mx\},$$

and let k be the cardinality of K.

We first construct a partition, $\{W_i\}_{i=1}^m$, of $X - P$ by l such that:

$$\frac{\mu_i(W_i)}{\mu_i(X-P)} > \frac{1/m}{\mu_i(X-P)} \quad \text{for } i \notin K,$$

and

$$\frac{\mu_i(W_i)}{\mu_i(X-P)} > \frac{1/m - Mx/k}{1-Mx} \quad \text{for } i \in K.$$

To do this, note that since

$$\sum_{i\notin K} \frac{1/m}{\mu_i(X-P)} + \sum_{i\in K} \frac{1/m - Mx/k}{1-Mx} < \sum_{i\notin K} \frac{1/m}{1-Mx} + \sum_{i\in K} \frac{1/m - Mx/k}{1-Mx} = 1,$$

there exist rational numbers $\{r_i\}_{i=1}^m$ such that

$$r_i > \frac{1/m}{\mu_i(X-P)} \text{ for } i \notin K, \quad r_i > \frac{1/m - Mx/k}{1-Mx} \text{ for } i \in K, \text{ and } \sum_{i=1}^m r_i < 1.$$

Lemma 7-3 shows the existence of a partition of $X - P$ by l, $\{W_i\}$, such that $\mu_i(W_i) \geq r_i$, and $\{W_i\}$ is a partition satisfying the desired inequalities above.

From Theorem 7-1, there is a partition, $\{V_i\}_{i=1}^k$, of P by l such that

$$\frac{\mu_i(V_i)}{\mu_i(P)} \geq \frac{1}{k} \quad \text{for } i \in K.$$

The partition, $\{U_i\}_{i=1}^m$, of X by l that satisfies the statement of the lemma is given by

$$U_i = W_i, \quad \text{for } i \notin K; \quad \text{and} \quad U_i = W_i \cup V_i, \quad \text{for } i \in K.$$

For $i \notin K$,

$$\mu_i(U_i) = \mu_i(W_i) \geq r_i\mu_i(X-P) > \frac{1/m}{\mu_i(X-P)}\mu_i(X-P) = \frac{1}{m}.$$

For $i \in K$,

$$\begin{aligned}\mu_i(U_i) = \mu_i(W_i) + \mu_i(V_i) &> \frac{1/m - Mx/k}{1-Mx}\mu_i(X-P) + \frac{1}{k}\mu_i(P) \\ &= \frac{1}{m} - \frac{Mx}{k} + \frac{1}{k}Mx. \quad \blacksquare\end{aligned}$$

THEOREM 7-5: *There is a partition, $\{U_i\}_{i=1}^m$, of X by l, so that either:* (i) *$\mu_i(U_j) = 1/m$ for all i and all j from 1 to m, or* (ii) *$\mu_i(U_i) > 1/m$ for all i.*

Proof: If the hypothesis of Lemma 7-4 is satisfied, then part (ii) is valid. Otherwise the partitioning process, used to prove Theorem 7-1, produces partitioning sets that all the measures μ_i assign measure $1/m$. ∎

COMMENT. If X is a general finite, non-atomic measure space (and not required to be a subset of a Banach space), then there is always a partition of X (not a partition by a continuous linear functional) by measurable sets that satisfies part (i) of the theorem. This follows from the Liapanov Theorem. For since $(1, 1, \ldots, 1)$ and $(0, 0, \ldots, 0)$ are in the range of $(\mu_1, \mu_2, \ldots, \mu_m)$, this range also contains $(1/m, 1/m, \ldots, 1/m) = (\mu_1(U_1), \mu_2(U_1), \ldots, \mu_m(U_1))$ for some measurable set U_1. Since $(\mu_1(X - U_1), \mu_2(X - U_1), \ldots, \mu_m(X - U_1)) = (1 - 1/m, 1 - 1/m, \ldots, 1 - 1/m)$, we can apply the Liapanov Theorem again to find a measurable set $U_2 \subseteq X - U_1$, such that $\mu_i(U_2) = 1/m$ for all i, and the partition is completed by induction.

In the general setting — unless all the measures, μ_i, are identical — there is always a partition so that part (ii) is true (Karlin and Studden [5]).

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